Discrete Mathematics 109 (1992) 297–306
North-Holland

Distance regular subgraphs of a cube

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Received October 1990
Revised 15 march 1991

Dedicated to Gert Sabidussi.

Abstract

Weichsel, P.M., Distance regular subgraphs of a cube, Discrete Mathematics 109 (1992) 297–306.

We examine the class of distance regular graphs which can be embedded in a cube. We show that in the case of isometric embedding they are precisely the cubes, the even cycles and the 'revolving doors'—a 'revolving door' is the subgraph of an odd dimensional cube whose vertices are as evenly balanced in the number of 1's and 0's as possible. In the general case we show that if the girth of the graph is 4, then it must be a cube, and we also obtain some bounds on the parameters of the graph.

1. Introduction

In this paper we examine those distance regular graphs which can be embedded isomorphically as subgraphs of hypercubes, hereafter referred to as 'cubes'. We consider two kinds of embeddings: *isometric* embeddings and *ordinary* embeddings. An embedding is called *isometric* if the distance separating a pair of vertices is the same in the original graph as in the cube. An *ordinary* embedding may not have this feature. Our results may be summarized as follows. (Recall that every subgraph of a cube is bipartite.) In the isometric case we obtain a characterization of these graphs.

We first need to define the *revolving door* D_k. Consider the vertices of the cube Q_{2k-1} which have k ones and $k-1$ zeros and those with $k-1$ ones and k zeros. The subgraph induced by these vertices is D_k.

Corollary 4. *Let Γ be a distance regular graph of valency k isometrically embedded in the cube Q_n, n minimal. Then either*:

(1) $\Gamma \simeq Q_n$ *or*

(2) $\Gamma \simeq C_{2r}$, *the* $2r$*-cycle with* $n = r$ *or*

(3) $\Gamma \simeq D_k$, *the revolving door with* $n = 2k - 1$.

Our results can also be stated in terms of girth.

Theorem 1. *Let Γ be a distance regular graph of girth $g = 4$ embedded in a cube. Then Γ is isomorphic to a cube.*

Theorem 2. *Let Γ be a distance regular graph isometrically embedded in a cube. If the girth of Γ is $g = 2r$ and $r > 3$, then Γ is an even cycle.*

Theorem 3. *Let Γ be a distance regular graph isometrically embedded in a cube. If the girth of Γ is* 6, *then either Γ is an even cycle or Γ is a revolving door.*

Notice that Theorem 3 provides a characterization of revolving doors.

In the non-isometric case we obtain some information about the parameters of Γ.

Theorem 7. *Let Γ be a distance regular graph of valency k, diameter d and girth at least* 6 *embedded in a cube. Then $d > k$.*

Theorem 5. *Let Γ be a distance regular graph of valency k, diameter d and girth at least* 6 *and embedded in a cube.* (*Thus Γ has intersection array* $\{k, b_1, \ldots, b_{d-1}; 1, 1, c_3, \ldots, c_d\}$.) (a) *if v is a vertex of Γ on level r and of weight r, then* $c_{r-1} \leq (2r-1)/3$. (b) *If $r \leq k$, then* $c_{r-1} \leq (2r-1)/3$.

2. Notation and definitions

Every connected graph can be represented by a *level diagram* as follows. Choose a vertex u and let it be the sole resident of level 0. The vertices on level i are precisely those whose distance from u is i. Now add the edges of the graph and note that edges occur only between vertices of adjacent levels and among vertices on the same level. In general the diagram will depend strongly on the choice of the level 0 vertex. Thus, if Γ is the 4 vertex line, we have the level diagrams in Fig. 1.

We can now define distance regularity of a graph Γ in terms of a level diagram. Let v be a vertex of Γ on level i and let level d be the last level.

(1) Denote by a_i the number of vertices on level i adjacent to v, $i = 1, \ldots, d$.

(2) Denote by b_i the number of vertices on level $i+1$ adjacent to v, $i = 0, 1, \ldots, d-1$.

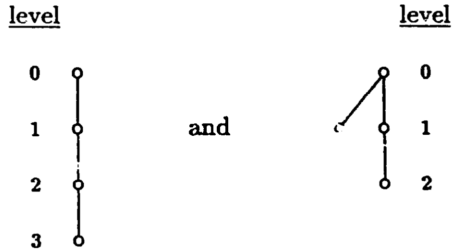

Fig. 1.

(3) Denote by c_i the number of vertices on level $i-1$ adjacent to v, $i=1, \ldots, d$.

The connected graph Γ is said to be *distance regular* if the numbers a_i, b_i, c_i, d do not depend on the choice of the level diagram or on the choice of v. This is easily seen to be equivalent to the more conventional definition of distance regularity as found, for example, in [2, Section 20].

(c_i has the following equivalent definition which we will have occasion to use in the sequel. If x and y are vertices a distance i apart, then c_i is the number of vertices adjacent to x which are a distance $i-1$ from y.)

It is clear that $b_0 = k$, the valency of Γ and $c_1 = 1$. Moreover, $a_i + b_i + c_i = k$ for $i = 1, \ldots, d$, if we impose the convention that $b_d = 0$. Also $c_{i+1} \geqslant c_i$ for all i. If the graph Γ is bipartite, then $a_i = 0$ for all i and $c_d = k$. Finally, if Γ has girth (length of shortest cycle) $2r$, then $c_1 = c_2 = \cdots = c_{r-1} = 1$ and $c_r > 1$. The array $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ is called the *intersection array* of Γ.

The level diagram can be very useful in studying a graph that is embedded in a cube. We will use the convention, apparently due to Rado, of defining the vertices of the cube Q_n as the subsets of $N = \{1, 2, \ldots, n\}$. The distance between a pair of vertices is then the cardinality of their symmetric difference. We will refer to the cardinality of the set representing a vertex as its *weight.*

If a graph Γ is embedded in a cube we can assume that ϕ is one of the vertices and we can draw a level diagram of Γ with ϕ at level 0. We will do this consistently throughout the paper. Then, if the embedding is isometric, each vertex on level i has weight i. If the embedding is ordinary, then the vertices on level 0, 1 and 2 have weights 0, 1 and 2 respectively, but a vertex on level r can have weight $r, r-2, r-4, \ldots, 2$ if r is even or $r, r-2, \ldots, 1$ if r is odd. Moreover, some graphs can have isometric embeddings and non-isometric embeddings as well. For example, if Γ is the 6-cycle, then we can have the situation in Fig. 2.

The first of these is isometric and the second is not. This behavior is not limited to cycles—every revolving door has non-isometric embeddings as well as isometric ones.

If Γ is bipartite of girth $g = 2r$, then it is easy to see that the level diagram of Γ is a tree from level 0 to level $r-1$. We name the vertices on level 1 by $1, 2, \ldots, k$

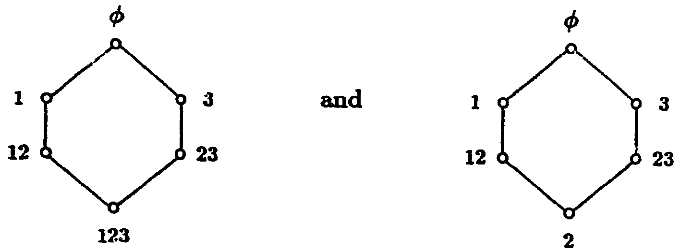

Fig. 2.

(we drop the distinction between 1 and $\{1\}$), where k is the valency of Γ and refer to the integers $\{1, 2, \ldots, k\}$ as the *primary* dimensions of Γ. Then the component of the vertex 1 in the forest obtained by deleting ϕ is called the *1-branch* of Γ and similarly for all other vertices on level 1.

Throughout the rest of the paper we will assume that the embeddable graph has a level diagram of the type described.

Remark. Our results may be considered from two different points of view. First, it may be seen as a contribution to the effort to characterize all subgraphs of a hypercube. Important contributions in this direction have been made by Garey and Graham [4], Djoković [3], Avis [1] and others but the general question remains open. Second, as the hypercubes are also distance regular graphs, our results give some insight into how distance regular graphs fit together. Nothing substantial is known about this question to the best of our knowledge.

3. Girth $g = 4$

The following lemma will be used throughout.

Lemma A. *Let Γ be a distance regular graph embedded in a cube. Suppose that there is a vertex v on level i which contains an integer not in any vertex on level $i-1$ which is adjacent to v. Then $c_i = 1$.*

Proof. Let $a \in v$ be the integer referred to in the statement. If w is adjacent to v and w is on level $i-1$, then $a \notin w$. Thus w consists of v with a removed and so w is unique. Hence $c_i = 1$. $\square$

Theorem 1. *Let Γ be a distance regular graph of girth $g = 4$ embedded in a cube. Then Γ is isomorphic to a cube.*

Proof. Assume that Γ is embedded in Q_n with n minimal. Suppose that $1, 2, \ldots, k$ are the vertices on level 1, $k \leq n$. If some integer $m > k$ occurs in some vertex on level $i > 1$, then let i be the first level on which this occurs. Then according to Lemma A, $c_i = 1$. But since the girth of Γ is 4, it follows that $c_2 > 1$ and thus $c_i > 1$ for all $i > 1$. Thus each vertex of Γ is a vertex of Q_k where k is the valency of Γ. Hence Γ is a subgraph which is regular of valency k of the cube Q_k which also has valency k. It follows that $\Gamma \simeq Q_k$. □

4. Isometric embeddings

In order to characterize those distance regular graphs which have an isometric embedding in a cube we need to examine the formal properties of the embedded vertices on the level diagram. Recall that if Γ is isometrically embedded in a cube, then vertices which are adjacent as vertices of the cube must also be adjacent as vertices of the graph.

Lemma B. *Let Γ be a distance regular graph isometrically embedded in a cube. Then every vertex of Γ in the tree of the level diagram has exactly one primary dimension.*

Proof. Suppose that v is a vertex on level i in the tree of the level diagram and assume that $1, 2 \in v$. Then, since the embedding is isometric the vertices 1 and 2 are both a distance $i - 1$ from v and adjacent to ϕ. Thus there are two paths from v to ϕ contradicting the fact that v is in the tree of the diagram. □

Lemma C. *Let Γ be a distance regular graph isometrically embedded in a cube. Suppose that the girth of Γ is $g = 2r$. Then $c_r = 2$.*

Proof. Since Γ has girth $2r$, $c_1 = c_2 = \cdots = c_{r-1} = 1$ and $c_r \geq 2$. Let u be a vertex on level r. It must be adjacent on level $r - 1$ to at least 2 vertices in different branches, since otherwise we would violate $g = 2r$. Thus assume v is adjacent to $\{1, a_1, \ldots, a_{r-2}\}$ and $\{2, b_1, \ldots, b_{r-2}\}$ with $a_i \neq 1, 2$ and $b_j \neq 1, 2$ for all i, j, where 1 and 2 are primary dimensions. But the symmetric difference of these two sets has size 2 since their distance apart is two. Hence $\{a_1, \ldots, a_{r-2}\} = \{b_1, \ldots, b_{r-2}\}$ and $v = \{1, 2, a_1, \ldots, a_{r-2}\}$. Now v cannot be adjacent to a vertex in any other branch. Thus $c_r = 2$. □

Lemma D. *Let Γ be a distance regular graph isometrically embedded in a cube. Let the girth of Γ be $g = 2r$. If $\{i, a_1, \ldots, a_{r-2}\}$ is a vertex on level $r - 1$ in branch i, then $\{j, a_1, \ldots, a_{r-2}\}$ is a vertex on level $r - 1$ in branch j, for each $j = 1, 2, \ldots, k$, where k is the valency of Γ.*

Proof. *Let $v = \{i, a_1, \ldots, a_{r-2}\}$* be a vertex in branch i on level $r - 1$. Then since $c_{r-1} = 1$, $b_{r-1} = k - 1$. Thus there are $k - 1$ vertices on level r which are adjacent to v. Each of these is also adjacent to another vertex on level $r-1$. By the argument in the proof of Lemma C, the vertex in branch j must have the form $\{j, a_1, \ldots, a_{r-2}\}$. Thus no two can be adjacent to vertices in the same branch and the result follows. □

Theorem 2. *Let Γ be a distance regular graph isometrically embedded in a cube. If the girth of Γ is $g = 2r$ and $r > 3$, then Γ is an even cycle.*

Proof. Let Γ have valency k. If $k = 2$, then Γ is an even cycle since Γ is bipartite. Thus assume that $k \geq 3$. Let $u = \{1, a_1, \ldots, a_{r-2}\}$ be a vertex on level $r-1$. Then there are vertices $v = \{2, a_1, \ldots, a_{r-2}\}$ and $w = \{3, a_1, \ldots, a_{r-2}\}$ on level $r-1$ by Lemma D. Thus there must be vertices of the form $\{1, 2, a_1, \ldots, a_{r-2}\}$, $\{1, 3, a_1, \ldots, a_{r-2}\}$ and $\{2, 3, a_1, \ldots, a_{r-2}\}$ on level r with the obvious adjacencies. But these six vertices form a 6-cycle, contradicting $r > 3$. Thus Γ is an even cycle. □

Theorem 3. *Let Γ be a distance regular graph isometrically embedded in a cube. If the girth of Γ is 6, then either Γ is an even cycle or Γ is a revolving door.*

Proof. If k, the valency of Γ is 2, then Γ is an even cycle. Thus assume that $k > 2$. It follows from Lemma D that if the vertices on level 1 are $1, 2, \ldots, k$, then the vertices on level 2 are of the form $\{i, j\}$ with $i = 1, \ldots, k$ and $j = k+1, \ldots, 2k-1$ and all such vertices occur. We now make the following induction assumption. The vertices on level $2s-1$ are all vertices of the form

$$\{i_1, \ldots, i_s, j_1, \ldots, j_{s-1}\}$$
$$i_l \in \{1, \ldots, k\} \quad \text{and} \quad j_m \in \{k+1, \ldots, 2k-1\}.$$

(When $s = 1$ no j's are present.) The vertices on level $2s$ are all vertices of the form $\{i_1, \ldots, i_s, j_1, \ldots, j_s\}$ where $\{i_1, \ldots, i_s, j_1, \ldots, j_{s-1}\}$ is a vertex on level $2s-1$ and $j_s \in \{k+1, \ldots, 2k-1\}$. Moreover, $c_{2s-1} = c_{2s} = s$.

The assumption is true for $s = 1$ and we proceed to consider levels $2s+1$ and $2s+2$.

As the embedding is isometric, there are two possibilities for a vertex on level $2s+1$: $\{i_1, \ldots, i_{s+1}, j_1, \ldots, j_s\}$ and $\{i_1, \ldots, i_s, j_1, \ldots, j_{s+1}\}$. If the second possibility occurs, then we will have the 4 vertices: $\{i_1, \ldots, i_s, j_1, \ldots, j_{s+1}\}$ on level $2s+1$, $\{i_1, \ldots, i_s, j_1, \ldots, j_s\}$ and $\{i_1, \ldots, i_s, j_1, \ldots, j_{s-1}, j_{s+1}\}$ on level $2s$ and $\{i_1, \ldots, i_s, j_1, \ldots, j_{s-1}\}$ on level $2s-1$ which clearly form a 4-cycle. This violates the assumption that the girth of Γ is 6. Thus all vertices on level $2s+1$ have the form $\{i_1, \ldots, i_{s+1}, j_1, \ldots, j_s\}$. Now since $c_{2s} = s$ and $b_{2s} = k - c_{2s} = k - s$, there are $k - s$ vertices on level $2s+1$ adjacent to each vertex on level $2s$. Hence all possible vertices of the form $\{i_1, \ldots, i_{s+1}, j_1, \ldots, j_s\}$ occur on level $2s+1$ and no others. But now each such vertex is adjacent to the $s+1$ vertices of the form $\{i_1, \ldots, i_s, j_1, \ldots, j_s\}$ on level $2s$ and so $c_{2s+1} = s+1$. Again $b_{2s+1} =$

$k - c_{2s+1} = k - s - 1$. Therefore each vertex of the form $\{i_1, \ldots, i_{s+1}, j_1, \ldots, j_s\}$ is adjacent to $k - s - 1$ vertices on level $2s + 2$. Now vertices on this level can have the form either $\{i_1, \ldots, i_{s+1}, j_1, \ldots, j_{s+1}\}$ or $\{i_1, \ldots, i_{s+2}, j_1, \ldots, j_s\}$. If the second possibility occurs, then we have the following 4 vertices: $\{i_1, \ldots, i_{s+2}, j_1, \ldots, j_s\}$ on level $2s + 2$, $\{i_1, \ldots, i_{s+1}, j_1, \ldots, j_s\}$ and $\{i_1, \ldots, i_s, i_{s+2}, j_1, \ldots, j_s\}$ on level $2s + 1$ and $\{i_1, \ldots, i_s, j_1, \ldots, j_s\}$ on level $2s$ which clearly form a 4-cycle violating the assumption that the girth of Γ is 6.

Thus only vertices of the form $\{i_1, \ldots, i_{s+1}, j_1, \ldots, j_{s+1}\}$ can occur on level $2s + 2$. But since each vertex on level $2s + 1$ is adjacent to $k - s - 1$ vertices on level $2s + 2$, it follows that level $2s + 2$ consists of all vertices of that form. Now it is clear that each such vertex is adjacent to precisely $s + 1$ vertices on level $2s + 1$ and so $c_{2s+2} = s + 1$.

Hence by induction we have a precise description of the vertices on all levels up to level $2k - 2$. The vertices on that level have the form $\{i_1, \ldots, i_{k-j}, j_1, \ldots, j_{k-1}\}$, since $c_{2s-2} = k - 1$, $b_{2k-2} = k - (k - 1) = 1$ and each such vertex is adjacent to $\{i_1, \ldots, i_k, j_1, \ldots, j_{k-1}\}$ which is unique, i.e., is equal to $\{1, 2, \ldots, k, k + 1, \ldots, 2k - 1\}$.

If we now interpret Γ as a set of vertices which are sequences of zeros and ones determined by their description as subsets and add the $2k - 1$ dimensional vertex

$$(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \bmod 2$$

to each vertex of Γ, then it is immediate that $\Gamma \simeq D_k$. $\square$

We can now summarize our results as follows.

Corollary 4. *Let Γ be a distance regular graph of valency k isometrically embedded in the cube Q_n, n minimal. Then either:*

(1) *$\Gamma \simeq Q_n$ or*

(2) *$\Gamma \simeq C_{2r}$, the $2r$-cycle with $n = r$ or*

(3) *$\Gamma \simeq D_k$, the revolving door with $n = 2k - 1$.*

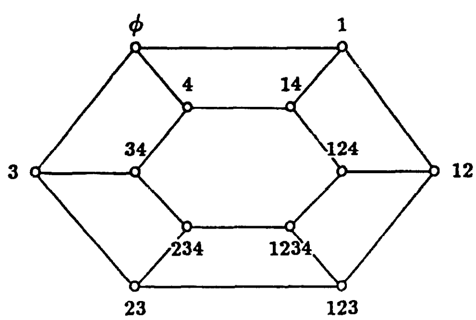

Fig. 3.

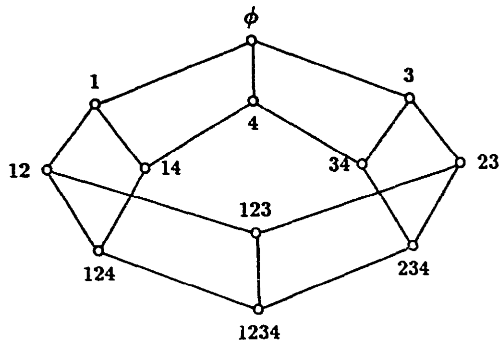

Fig. 4.

It is reasonable to ask whether the hypothesis that Γ is distance regular is necessary for the results obtained. Fig. 3 is an example of a vertex transitive but not distance regular graph Γ isometrically embedded in the cube Q_4. Its level diagram is shown in Fig. 4.

Remark. The class of all graphs which can be isometrically embedded in a cube were characterized by Djoković [3] in 1973. A variation of Djoković's characterization was formulated by Avis [1] a few years later. Our attempts to adapt these formulations to the case of distance regular graphs proved futile. It would be interesting to know if our characterization could be more readily obtained using either of these results.

5. The non-isometric case

In this section we present some results about those distance regular graphs which are embeddable, out not necessarily isometrically embeddable in a cube. Unfortunately we do not have an example of a distance regular graph which is embeddable but not isometrically embeddable in a cube. It is tempting to conjecture that there are not any.

In the next several results we examine a condition which always holds when the embedding is isometric, viz. a vertex on level r has weight r. The first result shows that the 'natural' progression of increasing weight with increasing level cannot terminate too high up in the level diagram.

Lemma E. *Let Γ be a distance regular graph of valency k embedded in a cube. Let v be a vertex on level r of weight r. If every vertex on level $r+1$ that is adjacent to v has weight $<r+1$, then $r \geq k$.*

Proof. Let v be the vertex in question. Every vertex adjacent to v on level $r-1$ is a subset of v of size $r-1$. The same is true about all vertices adjacent to v on level $r+1$. Thus the total number of such vertices is at most r. But that number is also the valency of v and hence $r \geq k$. □

Lemma F. *Let $\{a_1, \ldots, a_s\}$ be a collection of s subsets of size $n-1$ of the set $N = \{1, 2, \ldots, n\}$. Let A_i be the set of all $n-2$ element subsets of a_i. Then $|A_1 \cup \cdots \cup A_s| = (n-1)s - \binom{s}{2}$.*

Proof. Clearly $|A_i| = n-1$ for all i and $|A_i \cap A_j| = 1$ since $A_i \cap A_j$ consists of those $n-2$ element subsets from which the same 2 integers of N are missing. Clearly $A_i \cap A_j \cap A_l = \emptyset$. Thus by inclusion–exclusion,

$$|A_1 \cup \cdots \cup A_s| = (n-1)s - \binom{s}{2} \cdot 1. \quad \square$$

Theorem 5. *Let Γ be a distance regular graph of valency k, diameter d and girth at least 6 and embedded in a cube. (Thus Γ has intersection array $\{k, b_1, \ldots, b_{d-1}; 1, 1, c_3, \ldots, c_d\}$.)* (a) *If v is a vertex of Γ on level r and of weight r, then $c_{r-1} \leq (2r-1)/3$.* (b) *If $r \leq k$, then $c_{r-1} \leq (2r-1)/3$.*

Proof. The vertices adjacent to v on level $r-1$ are subsets of size $r-1$ of a set of size r and there are c_r of them. The vertices on level $r-2$ adjacent to these are subsets of size $r-2$ of v. If any of these sets are equal, then Γ has a 4-cycle, contrary to the hypothesis. By Lemma F there are exactly $(r-1)c_r - \binom{c_r}{2}$ such vertices. As each such vertex on level $r-1$ is adjacent to c_{r-1} vertices on level $r-2$, we have $c_r c_{r-1} \leq (r-1)c_r - \binom{c_r}{2}$. Simplifying we get $c_r + 2c_{r-1} \leq 2r-1$ and using the fact that $c_r \geq c_{r-1}$ we get $c_{r-1} \leq (2r-1)/3$. This proves (a).

Now if $r \leq k$, then by Lemma E there must be a vertex on level r that has weight r and (b) follows. □

A slightly weaker result can be obtained under the same hypotheses by using the inequality $c_r c_{r-1} \leq \binom{r}{2}$ which gives $c_{r-1} \leq \binom{r}{2}^{\frac{1}{2}}$. When the girth of Γ is larger than 6 we can easily extend the inequality and obtain the following result.

Theorem 6. *Let Γ be a distance regular graph of valency, k, diameter d and girth at least $2s$ and embedded in a cube. (Thus Γ has intersection array $\{k, b_1, \ldots, b_{d-1}; 1, \ldots, 1, c_s, \ldots, c_d\}$.) If $r \leq k$, then*

$$c_{r-(s-2)} \leq \binom{r}{s-1}^{1/(s-1)}.$$

Proof Since $r \leq k$, there is a vertex v on level r that has weight r. The set of vertices on level $r-(s-1)$ which are connected to v are all subsets of size $r-s+1$ of v. Since the girth of Γ is at least $2s$ no equalities occur, and so there are $c_r c_{r-1} \cdots c_{r-s+2}$ such vertices. But there are at most $\binom{r}{r-s+1}$ such subsets.

Thus $c_r c_{r-1} \cdots c_{r-s+2} \leq \binom{r}{r-s+1}$ and since $c_i \geq c_{i-1}$ we get

$$(c_{r-s+2})^{s-1} \leq \binom{r}{r-s+1} = \binom{r}{s-1},$$

$$c_{r-s+2} \leq \binom{r}{s-1}^{1/(s-1)}. \quad \square$$

Finally we show that the diameter of such an embedded graph cannot be small relative to the valency.

Theorem 7. *Let Γ be a distance regular graph of valency k, diameter d and girth at least 6 embedded in a cube. Then $d > k$.*

Proof. Suppose there is a vertex v on level d of weight d. Then the vertices $1, 2, \ldots, k$ on level 1 are a distance $d-1$ from v and adjacent to ϕ. It follows that $1, 2, \ldots, k \in v$ and so $d \geq k$. If $d = k$, then $v = \{1, 2, \ldots, k\}$ and so v is the unique vertex of weight k on level k. Since $c_k = k$, each subset of v of size $k-1$ occurs as a vertex on level $k-1$ adjacent to v. Denote them by $u_1, \ldots, u_k$. Each u_i is adjacent to c_{k-1} vertices on level $k-2$, each one a subset of size $k-2$ of v. Since Γ has no 4-cycles, there are exactly kc_{k-1} such vertices. Each u_i is also adjacent to b_{k-1} vertices on level k. Since v is the unique vertex of weight k on level k, each u_i is adjacent to $b_{k-1}-1$ vertices of weight $k-2$ on level k and since Γ has no 4-cycles, no two u_i's are adjacent to the same vertex of weight $k-2$. Thus there are exactly $k(b_{k-1}-1)$ such vertices. Now $kc_{k-1} + k(b_{k-1} - 1) = k(c_{k-1} + b_{k-1} - 1) = k(k-1)$ is the number of vertices of weight $k-2$ which are subsets of v. But there are only $k(k-1)/2$ such sets, a contradiction. Thus $k \neq d$.

Now assume that every vertex on level d has weight less than d. Then there is a level, say r, with $r < d$ containing a vertex v satisfying the hypothesis of Lemma E. Hence $r \geq k$ and so $d > k$. $\square$

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